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Solving the two-dimensional inverse heat source problem through the linear least-squares error method

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Abstract—An inverse model is presented for determining the strength of the temporal dependent heat source when the prior knowledge of the source functions are not available in the two-dimensional heat conduction problem. This model is constructed from the finite difference approximation of the differential heat conduction equation based on the assumption that the temperature measurements are available over the problem domain. In contrast to the traditional approach, the iteration in the proposed model can be done only once and the inverse problem can be solved in a linear domain. Three examples are used to show the usage of the proposed method. © 1997 Elsevier Science Ltd.

INTRODUCTION

The estimation of the boundary condition has received a great attention in the inverse heat conduction problem [1–8]. The inverse boundary problem is to estimate the boundary conditions from the knowledge of the temperature measurements taken at the interior point of the solid. They have extensive applications in many design and manufacturing problems in which the boundary condition is incomplete specification. However, there are only limited numbers of researches that work in the inverse heat source problem [9–10]. The inverse source problem is to estimate the strength of the heat source from the temperature measured at a different point other than the source's location. However, the problem has only been investigated with a one-dimensional inverse source problem, while the multi-dimensional problem still needs to be explored.

In the inverse analysis, most studies employed the nonlinear least-squares method [11–13] to determine the inverse problem. This method minimizes the formulation from the sum of the squares of the difference between the experimental measurements and the calculated response of the system. Based on the nonlinear least-squares method, various researchers have put their efforts in the field of inverse problems. In solving the problems, different algorithms have been adopted such as the conjugate gradient method, the Davidson–Fletcher–Powell method, the Monte-Carlo technique, the covariance analysis, and the dynamic programming. More sophisticated methods also have been developed such as the nonlinear least-squares formulation modified by the addition of a regularization term, the sequential estimation approach, and the adjoint equation approach coupled to the conjugate gradient method [1–7]. There are a few drawbacks in the above approaches. One is that the iterative process in the computation cannot be avoided. The other is that the inverse problem can only be solved in a nonlinear domain.

The purpose of this research is to propose an approach to replace the nonlinear least-squares method so that the iterative calculations in the analysis and optimization phases can be eliminated in the inverse heat source problem. In the proposed method, a linear inverse model is constructed to represent the undetermined heat source explicitly.

In the process of constructing the linear model, the strength of the heat source is represented by the series form first. Then, the series form of the heat source and the approximation form of the heat conduction equation are rearranged. Finally, the available temperature field are substituted into the approximation model. As a result, the approximation model becomes a linear combination of the unknown coefficients for the strength of the heat source and then this linear inverse model can lead to solving the problem through the linear least-squares error method. The advantage of this approach is that the computation in the process can be done only once and the inverse problem can be solved in a linear domain. In this paper, only the linear case is considered. That means there are no temperature-dependent coefficients in the heat equation or in the boundary conditions. In the nonlinear problems, the present analysis can be used to compute the associated linearized equations. Furthermore, it is not difficult to extend our analysis to the various geometries in spatial domains.

DESCRIPTION OF THE PROPOSED METHOD

Consider an infinitely long bar with constant thermal properties and with a square-cross-section in

NOMENCLATURE

A	stiffness matrix of heat equation
b	vector of source function
a	undetermined variable
s	estimated heat source function
S	coefficient matrix of **b**
T	temperature vector
T	temperature
t	temporal coordinate
x, y	spatial coordinate.

Greek symbols

ψ	basis function
Δx, Δy	increment of spatial coordinate
Δt	increment of temporal domain
$\boldsymbol{\theta}$	coefficient vector of **b**, solution vector
λ	random number
ω	bound of random number
$\bar{m}$	upper bound of the index of coefficients.

Subscripts

i, j, k	indices
exact	exact temperature
m	index of undetermined coefficients
null	non-measured matrix
measurement	measurement temperature.

Superscripts

i, j, k	indices
meas	measured temperature vector
null	non-measured matrix.

which each side is one unit (see Fig. 1). The adiabatic conditions are applied at the side of $x = 0$ and $y = 1$. The isothermal conditions are applied at the side $x = 1$ and $y = 0$. It is initially at a uniform temperature T_0 and then suddenly a heat source function $s(t)$ is applied at $x = \bar{x}$ and $y = \bar{y}$. A dimensionless mathematical formation of the two-dimensional heat conduction problem is presented as follows:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+s(t)\delta(x-\bar{x}, y-\bar{y}) = \frac{\partial T}{\partial t}$$

$$0 < x < 1, \quad 0 < y < 1 \quad t > 0 \quad (1)$$

$$T(x, y, 0) = T_0 = 1 \quad 0 \leqslant x \leqslant 1 \quad 0 \leqslant y \leqslant 1 \quad t = 0 \quad (2)$$

$$T(1, y, t) = 1 \quad x = 1 \quad 0 \leqslant y \leqslant 1 \quad t > 0 \quad (3)$$

$$T(x, 0, t) = 1 \quad 0 \leqslant x < 1 \quad y = 0 \quad t > 0 \quad (4)$$

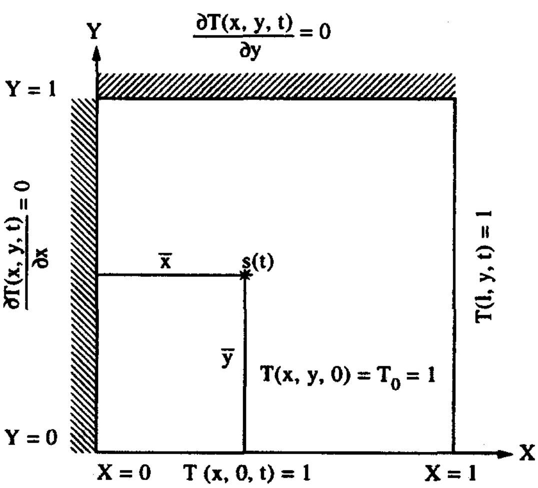

Fig. 1. A square cross-section of an infinitely long bar.

$$\frac{\partial T(x, y, t)}{\partial x} = 0 \quad x = 0 \quad 0 \leqslant y < 1 \quad t > 0 \quad (5)$$

$$\frac{\partial T(x, y, t)}{\partial y} = 0 \quad 0 \leqslant x < 1 \quad y = 1 \quad t > 0 \quad (6)$$

where $s(t)$ is the strength of the heat source and $\delta(x-\bar{x}, y-\bar{y})$ is Dirac delta function.

The inverse problem is to identify the strength of the heat source $s(t)$ from the temperature measurements taken at one of the interior points of the bar. Suppose that the applied heat source term $s(t)$ are represented in the following series form in a certain time domain:

$$s(t) = \sum_{m=1}^{\bar{m}} a_m \psi_m(t) \quad (7)$$

where $\psi_m(t)$ can be any non-singular function in the problem domain, a_m is the coefficients, and $\bar{m}$ is a positive integer.

For illustration, the implicit finite-difference methods is used to execute the analysis process. After discretization, the above governing equation is combined with $s(t)$ and can be expressed as the following recursive form:

$$\frac{1}{\Delta x^2}(T_{i-1,j,k}-2T_{i,j,k}+T_{i+1,j,k}) + \frac{1}{\Delta y^2}(T_{i,j-1,k}-2T_{i,j,k}+T_{i,j+1,k}) + s(t_k)\delta(x-\bar{x}, y-\bar{y}) = \frac{1}{\Delta t}(T_{i,j,k}-T_{i,j,k-1}) \quad (8)$$

where Δx and Δy are the increment of the spatial-coordinate and Δt is the increment of the temporal-coordinate, i is the ith grid along with x-coordinate, j is the jth grid along with y-coordinate, k is the kth grid along with temporal-coordinate, and $T_{i,j,k}$ is the temperature at the grid point (i, j, k).

Using the recursive form a matrix equation for the direct analysis can be expressed as

$$\mathbf{AT} = \mathbf{b} \tag{9}$$

where **A** is a stiffness matrix. It is constructed from the thermal properties, the spatial-coordinate, the temporal-coordinate, the boundary conditions and the initial condition. The components of **T** are the unknown temperature in the discretized points, and the components of **b** are the functions of the boundary condition, the initial condition, and the source term.

In the inverse analysis, **A** can be constructed according to the known physical model and numerical methods and **T** can be measured by the thermocouples. In this problem, the boundary condition and the initial condition are known and the coefficients of $s(t)$ is the main task to be resolved. Decoupling the coefficients of $s(t)$ from **b** will transfer the direct formulation to the following form:

$$\mathbf{AT} = \mathbf{S}\boldsymbol{\theta} \tag{10}$$

where $\mathbf{b} = \mathbf{S}\boldsymbol{\theta}$. **S** is the coefficient matrix of **b** and $\boldsymbol{\theta}$ is the coefficient vector of **b**. **S** is a constant matrix. The components of $\boldsymbol{\theta}$ are the functions of the unknown variables a_m in equation (7).

The inverse matrix of **A** is multiplied into both sides of equation (10), we have

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{S}\boldsymbol{\theta} = \mathbf{C}\boldsymbol{\theta} \tag{11}$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{S}$ and **T** is the temperature vector in the bar. In the inverse analysis, there are only a few components of **T** needed to be measured in order to identify the strength of the heat source. Therefore, only vector $\mathbf{T}^{\text{meas}}$ and matrix **D** that correspond with the measured grids need to be constructed. Thus, equation (11) can be shown as follows:

$$\begin{bmatrix} \mathbf{T}^{\text{meas}} \\ \mathbf{T}^{\text{null}} \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}_{\text{null}} \end{bmatrix} \boldsymbol{\theta} \tag{12}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}^{\text{meas}} \\ \mathbf{T}^{\text{null}} \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}_{\text{null}} \end{bmatrix}$$

$\mathbf{T}^{\text{null}}$ and $\mathbf{D}_{\text{null}}$ are the part of matrices of **T** and **C**; they correspond with the non-measured grids.

The relationship between the measured temperature $\mathbf{T}^{\text{meas}}$ and the undetermined coefficients $\boldsymbol{\theta}$ can be represented in the following form:

$$\mathbf{T}^{\text{meas}} = \mathbf{D}\boldsymbol{\theta}. \tag{13}$$

Then, $\boldsymbol{\theta}$ can be solved by the linear least-squares error method as follows:

$$\boldsymbol{\theta} = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{T}^{\text{meas}} \tag{14}$$

$(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}$ is the reverse matrix of the undetermined heat source and is denoted as **R**.

In equation (14), the inverse problem is solved by the linear least-squares error method. As such, the iterative process in the problem can be avoided, and the problem is solved in a linear domain. Furthermore, it can be verified that the final solution, i.e. equation (14), from the proposed method is the necessary condition of the optimum from the traditional nonlinear least-squares approach [10]. In the inverse problem, it is important to investigate the stability of the estimation. Usually, a minor measurement error makes the estimation away from the exact solution in the ill-posed inverse problem. The methods of future time and regularization have been widely used to stabilize the results of the inverse estimation [2, 3]. Those methods impose the physical condition onto the problem and increase the computational load in the estimated process. Consequently, the stability of the problem can be increased, while the computational load of the problem is also increased. In the present research, it is possible to stabilize the estimated results through a smooth process [14]. This method computes a 'moving average' of the estimation. The results of data is the average of the N-point around the current point. In this process, N must be an odd number. Then, the efficiency of the estimation can be arisen.

In solving equation (14), the number of measurements needs to be sufficient so that the rank of the reverse matrix is equal to the number of undetermined coefficients. Otherwise, equation (14) will be underdetermined and the problem cannot be solved through the proposed method. In general, when a large amount of the measurements are selected, the cost for computation and experiment increase. Yet, the accuracy of the estimated results increase as well. Furthermore, it is possible to recognize the existence and uniqueness of the solution when the rank of the reverse matrix is equal to the number of the unknown variables. If the matrix equation [equation (13)] is consistent (i.e. the measurement errors are not considered), the solution exists and is unique. If the matrix equation is inconsistent, a unique least-squares solution can be approximated.

RESULTS AND DISCUSSION

In this section, problems defined from equation (1)–(6) are used as examples for the proposed method for estimating the strength of the heat source. We have chosen a stepwise variation over time for the source strength in the first example, a triangular variation in the second example, and a sinusoidal–exponential–polynomial function in the third example. The first example is used to demonstrate the uniqueness of the solution through the numerical values when the measured points are at the same distance from the source location and the same condition from the boundaries. The second example is used to compare the accuracy and robustness of the estimated results when the measured points are at different distances from the source location. The third example is used to demonstrate the accuracy and robustness of the estimated results when measurement errors are increased. The exact temperature and source term used in the following examples are preselected so that these functions can satisfy equation (1)–(6). The accuracy of the proposed method is assessed by comparing the estimated results with the preselected strength value of the heat source. Meanwhile, the measured temperature is generated from the preselected exact temperature in each problem and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equations:

$$T_{\text{measurement}} = T_{\text{exact}} + \lambda T_{\text{exact}} \tag{15}$$

and

$$|\lambda| \leqslant \omega \tag{16}$$

where λ is the random error of measurement, and ω is the bound of the λ. T_{exact} in equation (15) is the exact temperature. $T_{\text{measurement}}$ is the measured temperature at the grid points and it is the component of the vector $\mathbf{T}^{\text{meas}}$.

The time domain in all cases is from 0 to 0.3 with 0.01 increments and the increment of spatial coordinates are also 0.1. When the prior knowledge of the source functions are not available, the estimated heat sources need to be expressed by 30 coefficients ($\bar{m} = 30$) and each coefficient represents the heat source at a specific temporal-grid in each example. For example, a_1 is the value of $s(t)$ at $t = 0.01$, a_2 is the value of $s(t)$ at $t = 0.02$, a_m is the value of $s(t)$ at $t = t_m = 0.01 \times m$ and so on. In the examples, the value of $\psi_m(t)$ is equal to one when $t = t_m$. When $t \neq t_m$, the value of $\psi_m(t)$ is vanished. The estimated results of the source strength from the knowledge of the temperature at measurement points are examined. The estimated results are also smoothed and the number of the surrounding points is three in all examples. As a result, when measurement errors are not considered, the results have excellent approximations. It shows that the estimated result is converged to the exact solution in all examples. Furthermore, it is confirmed that the solution exists and it is unique through the verification of the proposed method. However, when measurement errors are included, the estimated results are deviated from the exact solution. Through the proposed method, it is proved that the unique least-square solutions exist in all examples. The following three examples demonstrate the application of the proposed approach. Detailed descriptions for the examples are shown as follows:

Example 1

Consider a bar with $L = 1$ (see Fig. 1). A stepwise variation of the strength of the heat source located at $(\bar{x} = 0.5, \bar{y} = 0.5)$ is defined as:

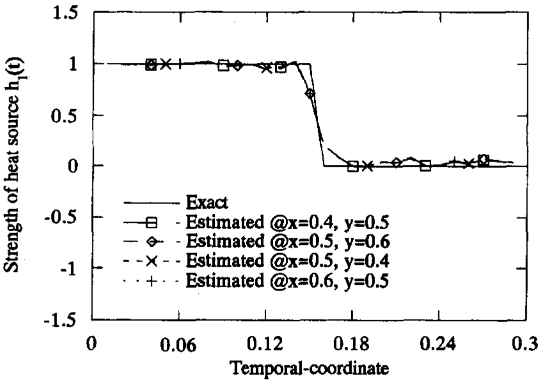

Fig. 2. Estimation of the stepwise heat source $h_1(t)$ on four different measured positions that have the same distance from the source location in example one (measurement errors $\omega = 1\%$).

$$h_1(t) = 1 \quad 0 < t \leqslant 0.15$$
$$h_1(t) = 0 \quad 0.15 < t \leqslant 0.3 \tag{17}$$

which represents a stepwise variation over time. In the first example, one measured point is allocated at $(x = 0.4, \ y = 0.5)$, $(x = 0.5, \ y = 0.4)$, $(x = 0.5, \ y = 0.6)$, or $(x = 0.6, y = 0.5)$.

The estimated results are shown in Fig. 2 for 1% measurement errors and in Fig. 3 for 2% measurement errors. In general, large errors make the estimated results away from the exact solution. The results shown in Fig. 3 ($\omega = 2\%$) have much larger deviations from the exact solution than those in Fig. 2 ($\omega = 1\%$). From the numerical results shown in Figs 2 and 3, the estimated results from the measured points at $x = 0.4$, $y = 0.5$ and $x = 0.5$, $y = 0.6$ are the same. Furthermore, the estimated results from the measured points at $x = 0.5, y = 0.4$ and $x = 0.6, y = 0.5$ are the same. The reason for that is that the four measured points are at the same distance from the source point, and each pair of measured points (i.e. $x = 0.4$, $y = 0.5$ and $x = 0.5$, $y = 0.6$; $x = 0.5$, $y = 0.4$ and $x = 0.6$, $y = 0.5$) have the same condition from the boundaries.

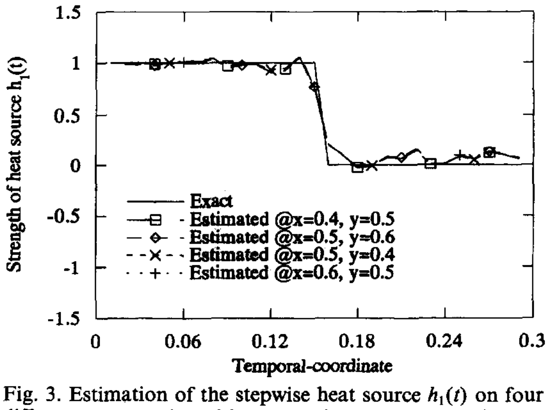

Fig. 3. Estimation of the stepwise heat source $h_1(t)$ on four different measured positions that have the same distance from the source location in example one (measurement errors $\omega = 2\%$).

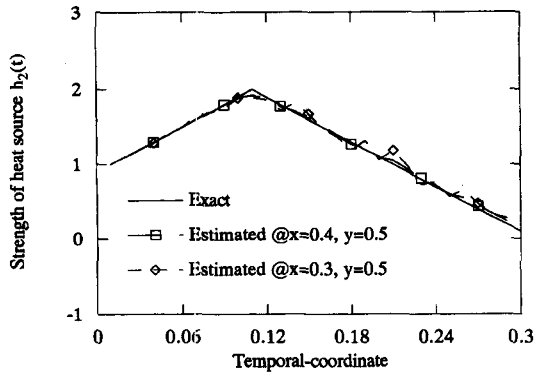

Fig. 4. Estimation of the triangular heat source $h_2(t)$ on two measured positions that have the different distance from the source location in example two (measurement errors $\omega = 1\%$).

Example 2

The second example is the same as example one, except that the source strength is expressed in a different form and it can be shown as follows:

$$h_2(t) = 10t+0.9 \quad 0 < t \leqslant 0.11$$

$$h_2(t) = -10t+3.1 \quad 0.11 \leqslant t \leqslant 0.3 \qquad (18)$$

which represents a triangular variation over time. In the second example, one thermocouple is allocated either at the measured point $x = 0.4$, $y = 0.5$ or at another point $x = 0.3$, $y = 0.5$.

The estimated results are shown in Fig. 4 for 1% measurement errors and in Fig. 5 for 2% measurement errors. From the numerical results in Figs 4 ($\omega = 1\%$) and 5 ($\omega = 2\%$), the estimated results from the measured point $x = 0.4$, $y = 0.5$ have better estimations than those from the measured point $x = 0.3$, $y = 0.5$. It appears that a closer measured position to the source location needs to be taken in order to have a more accurate and robust estimation.

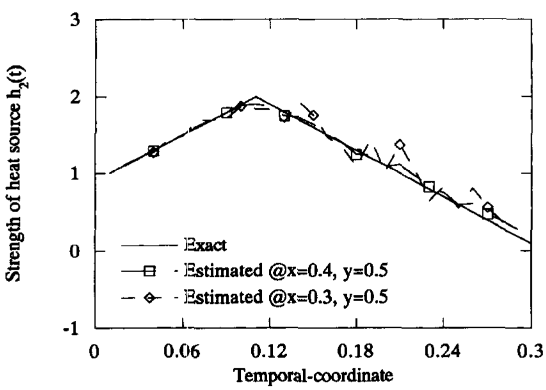

Fig. 5. Estimation of the triangular heat source $h_2(t)$ on two measured positions that have the different distance from the source location in example two (measurement errors $\omega = 2\%$).

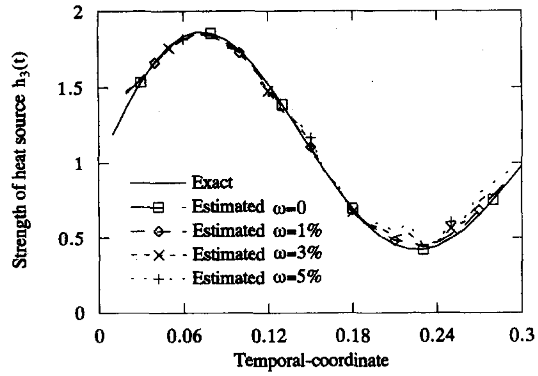

Fig. 6. Estimation of the heat source $h_3(t)$ measured at ($x = 0.4$, $y = 0.5$) when measurement errors are $\omega = 0$, 1, 3 and 5%.

Example 3

The third example is the same as that in example 1, except that the source strength is expressed in the following form:

$$h_3(t) = \sin\left(\frac{2\pi t}{0.31}\right)\times e^{-2t}+t^2+1 \quad 0 < t \leqslant 0.3. \qquad (19)$$

The estimated results for the measured point at $x = 0.4$, $y = 0.5$ are shown. In Fig. 6, the estimated results are approximated to the exact solution when measurement errors are not considered. In the same figure, it is also shown that the estimated results are accurate and robust when measurement errors are included. Moreover, when the measurement error is 5%, the result is still satisfied.

CONCLUSION

An inverse method has been introduced for determining the unknown strength of the heat source in a two-dimensional inverse conduction problem. The proposed inverse model is constructed from the available temperature measurements and the finite difference model of the differential heat conduction equation. This model can represent the undetermined strength of the heat source explicitly. Three examples have been illustrated using the proposed method. In the first example, the uniqueness of the solution can be verified numerically when the measured points are at the same distance from the source location and the same condition from the boundaries. In the second example, the results show when the measured point is closer to the source location, the properties of the estimated results are more accurate and robust. In the third example, the estimated results are accurate and robust even when measurement error is 5%. The advantage of this method is that the strength of the heat source can be estimated directly and the inverse problem can be solved in a linear domain. It is different from the traditional method using nonlinear least-squares formulation, which requires numerous iterations in the process and needs to perform its calculation in the nonlinear domain. The proposed method is applicable to other kinds of inverse problems such as initial estimation and boundary estimation in the one- or multi-dimensional heat transfer problems.

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